Derivation of a Squared Ellipsoidal Lobe Function

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1 Squared Spheroidal Lobe (SSL)

To derive a squared ellipsoidal lobe (SEL) function, we start from the following squared spheroidal lobe (SSL):

\[ \pi \alpha^2 D \left( \cos \frac{\theta}{2}, \hat{\alpha} \right) = \frac{4\hat{\alpha}^4}{(1 - \cos \theta + \hat{\alpha}^2(1 + \cos \theta))^2}. \]

where \( \theta \) is the angle between a direction \( \omega \in S^2 \) and the lobe axis \( \omega_0 \in S^2 \), \( \hat{\alpha} \in [0, 1] \) is the roughness of the lobe, and \( D(\cos \theta, \hat{\alpha}) \) is the isotropic GGX distribution [TR75, WMLT07]. Tokuyoshi and Harada [TH17] derived \( \sqrt{\pi} \alpha^2 D \left( \cos \frac{\theta}{2}, \hat{\alpha} \right) \) is a spheroid whose center and semiaxes in the lobe space are \( [0, 0, \frac{1 - \alpha^2}{2}] \) and \( [1, 1, \frac{1 + \alpha^2}{2}] \), respectively. Therefore, the lobe-space center \( c \) and semiaxes \( r \) of \( \sqrt{\pi} \alpha^2 D \left( \cos \frac{\theta}{2}, \hat{\alpha} \right) \) are given by

\[
\begin{align*}
\mathbf{c} &= \left[ 0, 0, \frac{1 - \alpha^2}{2} \right], \\
\mathbf{r} &= \left[ \hat{\alpha}, \hat{\alpha}, \frac{1 + \alpha^2}{2} \right].
\end{align*}
\]

2 Extension to a Squared Ellipsoidal Lobe (SEL)

This paper extends semiaxes \( r \) using anisotropic roughness parameters \( [\hat{\alpha}_x, \hat{\alpha}_y] \) as follows:

\[
\mathbf{r} = \left[ \hat{\alpha}_x, \hat{\alpha}_y, \frac{1 + \hat{\alpha}_{\text{max}}^2}{2} \right],
\]

where \( \hat{\alpha}_{\text{max}} = \max(\hat{\alpha}_x, \hat{\alpha}_y) \). For this, the lobe-space center is

\[
\mathbf{c} = \left[ 0, 0, \frac{1 - \hat{\alpha}_{\text{max}}^2}{2} \right].
\]

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Our SEL function is given by the squared distance from the origin to this ellipsoid. Therefore, we derive the SEL using the intersection of this ellipsoid and a line. A position on this line is given by

\[ p = t \omega, \]

where \( t \) is a distance from the origin. The ellipsoid-line intersection is equivalently rewritten into the intersection of a transformed line and a unit sphere centered at the origin. For this, a position on this line is given by

\[ p' = (t \omega - c) \begin{bmatrix} 1 \frac{1}{d_x} & 0 & 0 \\ 0 & 1 \frac{1}{d_y} & 0 \\ 0 & 0 & 2 \frac{2}{1 + d_z^2} \end{bmatrix} = td + s, \]

where \( \omega = [x, y, z] \). The intersection point of this line and the unit sphere is given as

\[ \|p'\|^2 = 1. \]

It is rewritten into a quadratic equation:

\[ \|d\|^2 t^2 + 2(d \cdot s)t + \|s\|^2 - 1 = 0. \]

The positive solution of this equation is given by

\[ t = \sqrt{(d \cdot s)^2 - \|d\|^2(\|s\|^2 - 1)} - d \cdot s. \tag{3} \]

Substituting Eq. (1) and Eq. (2) into Eq. (3), the solution is obtained as follows:

\[ t = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 + \frac{2}{1 + d_z^2}(1 - d_z^2)} \]

\[ = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 + \frac{2}{1 + d_z^2}(1 - d_z^2)} \]

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\[ = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 - \frac{2}{1 + d_z^2}(1 - d_z^2)} \]

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\[ = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 - \frac{2}{1 + d_z^2}(1 - d_z^2)} \]

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\[ = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 - \frac{2}{1 + d_z^2}(1 - d_z^2)} \]

\[ = \frac{1 + d_z^2}{2} \sqrt{\frac{\partial s_{\max}}{\partial x^2} x^2 + \frac{\partial s_{\max}}{\partial y^2} y^2 + z^2 - \frac{2}{1 + d_z^2}(1 - d_z^2)} \]
Therefore, the SEL is derived as

\begin{equation}
K(\omega; E, \hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z) = t^2 = \frac{4\alpha_{\text{max}}^4}{\left(1 + \alpha_{\text{max}}^2\right) \sqrt{\frac{d_{\text{max}}^2}{d_{\hat{\alpha}_x}^2} x^2 + \frac{d_{\text{max}}^2}{d_{\hat{\alpha}_y}^2} y^2 + z^2 - z \left(1 - \alpha_{\text{max}}^2\right)}}^2.
\end{equation}

where \( E \) is the 3×3 identity matrix. To represent the orientation of the lobe, Eq. (2) is generalized using a 3×3 orthogonal matrix \( Q \) as follows:

\begin{equation}
K(\omega; Q, \hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z) = \frac{4\alpha_{\text{max}}^4}{\left(1 + \alpha_{\text{max}}^2\right) \sqrt{\frac{d_{\text{max}}^2}{d_{\hat{\alpha}_x}^2} v_x^2 + \frac{d_{\text{max}}^2}{d_{\hat{\alpha}_y}^2} v_y^2 + v_z^2 - v_z \left(1 - \alpha_{\text{max}}^2\right)}}^2.
\end{equation}

where \([v_x, v_y, v_z]^T = Q\omega^T\) is the direction transformed into the lobe space. This SEL can also be rewritten into the following form:

\begin{equation}
K(\omega; Q, \hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z) = \frac{4\alpha_{\text{max}}^4}{\left((U - v_z) + \alpha_{\text{max}}^2 (U + v_z)\right)^2}.
\end{equation}

where \( U = \sqrt{\frac{d_{\text{max}}^2}{d_{\hat{\alpha}_x}^2} v_x^2 + \frac{d_{\text{max}}^2}{d_{\hat{\alpha}_y}^2} v_y^2 + v_z^2}\).

References

